Conference «Science of the Future», Mathematical section
St.Petersburg , September 17 – 20 , 2013

ALAG:
ABELIAN LAGRANGIAN ALGEBRAIC GEOMETRY

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Plan

1. Geometric pre-quantization equipment
2. Bohr – Sommerfeld cycles, Planckian cycles, and Berry bundle
3. Half-weights and BPU-mapping
4. Example: moduli space of rank 2 vector bundles on Riemann surface

Pre-quantized symplectic manifold

We consider data set $(M, \omega, I, L, \alpha)$ that consist of:

1. real smooth manifold M of dimension $\dim_{\mathbb{R}} M = 2n$
2. smooth closed 2-form $\omega \in \mathcal{C}^{\infty}(M, \Lambda^2 T^* M)$ that gives an isomorphism $\omega : TM \Rightarrow T^* M$
3. complex structure (possibly non-integrable) $I : TM \to TM$, $I^2 = -\mathrm{Id}$, such that $\omega(Iv, Iw) = \omega(v, w)$ and $g(v, w) \stackrel{\text{def}}{=} \omega(Iv, w)$ is positive
4. smooth Hermitian line bundle L over M such that $[\omega] = 2\pi c_1(L) \in 2\pi \cdot H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})$. We write $P \subset L$ for the principal U_1-bundle of unit circles in L and $u = 2\pi\, \partial/\partial t$ for vertical U_1-invariant vector field on P whose flow turns round about a fiber of P in the unit time.
5. Hermitian connection $\alpha \in \mathcal{C}^{\infty}(P, T^* P) : \mathcal{L}ie_u \alpha = 0,\ \alpha(u) = 1$. It provides L with covariant differentiation of local sections over open $\mathcal{U} \subset X$ $\nabla_\alpha : \mathcal{C}^{\infty}(\mathcal{U}, L) \to \mathcal{C}^{\infty}(\mathcal{U}, T^* M \otimes L)$, which takes an unitary section $\sigma : \mathcal{U} \hookrightarrow P \subset L$ to $\nabla_\alpha \sigma \stackrel{\text{def}}{=} i\xi_\sigma \otimes \sigma$, where $\xi_\sigma \stackrel{\text{def}}{=} \sigma^* \alpha \in \mathcal{C}^{\infty}(\mathcal{U}, T^* X)$. Thus, the compatibility 4 $\Rightarrow d\xi_\sigma = \omega$.

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Digression: on flat hermitian connections

Given a Hermitian line bundle L on arbitrary smooth variety X, a Hermitian connection α on L is called *flat* if $\omega_\alpha \stackrel{\text{def}}{=} d\xi_\sigma = 0$. Associated with flat α are:

- conjugation class of character $\chi_\alpha : \pi_1(X) \to U_1$ that sends a loop γ to the rotation of fiber of L at a base point under the horizontal displacement along γ (horizontality of section $f\sigma \iff d\ln f = -i\xi_\sigma$)
- cohomology class $[\chi_\alpha] \in H^1(X, U_1)$ represented by a Čech cocycle provided by transition functions between local horizontal unitary trivializations of L; it is annihilated by the red arrow in
$$H^1(X,\mathbb{Z}) \hookrightarrow H^1(X,\mathbb{R}) \to H^1(X,U_1) \to H^2(X,\mathbb{Z})$$
coming from the exponential exact triple of coefficients
$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{t \mapsto \exp 2\pi i t} U_1 \to 1 .$$

Equivalence classes of flat hermitian connections on L w.r.t. the action of gauge group $C^\infty(X, U_1)$ stay in bijection with the conjugation classes of unitary characters $\pi_1(X) \to U_1$ and with the points of the real Jacobian torus
$$J_X \stackrel{\text{def}}{=} H^1(X,\mathbb{R})/2\pi \cdot H^1(X,\mathbb{Z}).$$

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Lagrangian cycles

Fix smooth real n-dimensional compact oriented manifold S and a homotopy class $\mathrm{Map}_{\mathrm{LA}}(S,M)$ of smooth maps $\varphi : S \to M$ such that $\varphi^*(\omega) = 0$. Factor space

$$\mathfrak{L} \stackrel{\text{def}}{=} \mathrm{Map}_{\mathrm{LA}}(S,M)/\mathrm{Diff}_0(S)$$

is called a *space of Lagrangian cycles* (of fixed topological type).

Basic example: cotangent bundle

Let $M = T^*S$, $\omega = d\eta$, where η is the universal 1-form such that

$$\forall \xi \in C^\infty(S, T^*S) \quad \xi = s_\xi^* \eta,$$

where $s_\xi : S \hookrightarrow T^*S$ is the section provided by ξ. The fibers of $T^*S \twoheadrightarrow S$ and the zero section $s_0 : S \hookrightarrow T^*S$ are Lagrangian. Arbitrary section $s_\xi : S \hookrightarrow T^*S$ given by 1-form $\xi = s^*\eta$ is Lagrangian iff ξ is closed, because

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Darboux – Weinstein uniformisation of $\mathfrak{L}$

Claim

Near a smooth immersion $\varphi : S \hookrightarrow M$ the space of Lagrangian cycles $\mathfrak{L}$ is an (infinite dimensional) smooth manifold locally modeled by some open neighborhood $\mathfrak{U}$ of the zero in a vector space $Z^1_{\mathrm{DR}}(S)$ of closed 1-forms on S.

Classical results of A. Weinstein show that

- Any smooth immersion $\varphi : S \hookrightarrow M$ can be extended to a smooth immersion $\widehat{\varphi} : \mathcal{U} \hookrightarrow M$ of some open tube neighbor $\mathcal{U} \subset T^*S$ about the zero section of the cotangent bundle in such a way that $\widehat{\varphi}^*\omega = d\eta$ is the standard symplectic form on T^*S.
- Let $\mathfrak{U} \subset Z^1_{\mathrm{DR}}(S) = T_S\mathfrak{L}$ be the set 1-forms α on S whose graphs $\Gamma_\alpha \subset T^*S$ lie in $\mathcal{U}$. Then the previous extension leads to a smooth exponential mapping $\exp : \mathfrak{U} \to \mathfrak{L}$ sending such a form α to Lagrangian cycle $\widehat{\varphi}(\Gamma_\alpha) \subset M$ laying near $\varphi(S)$.
- The image of the exponential mapping contains an open neighborhood of φ in appropriate topology on $\mathfrak{L}$.

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Bohr – Sommerfeld cycles

Pull back $\varphi^* L$ of pre-quantization bundle along any Lagrangian immersion $\varphi : S \hookrightarrow M$ is a flat Hermitian line bundle on S, because of $\omega_{\varphi^*\alpha} = \varphi^*\omega = 0$.

Definition

Immersed Lagrangian cycle $\varphi : S \hookrightarrow M$ is called *Bohr – Sommerfeld* if $\varphi^* L$ admits a *global horizontal trivialization*, i.e. $[\chi_{\varphi^*\alpha}] \in J_S = H^1(S,\mathbb{R})/H^1(S,\mathbb{Z})$ vanishes. We write $\mathfrak{B} \subset \mathfrak{L}$ for the locus of Bohr – Sommerfeld cycles.

Thus, $\mathfrak{B} \subset \mathfrak{L}$ is the zero set of section $\varphi \mapsto [\chi_{\varphi^*\alpha}]$ of the trivial Jacobian bundle $\mathfrak{L} \times J_S$ over $\mathfrak{L}$ with fiber J_S and has finite expected codimensinon

$$\text{ex.codim}_{\mathfrak{L}}\mathfrak{B} = b_1(S) = \dim H^1(X,\mathbb{R}).$$

We expect a finite number of Bohr – Sommerfeld cycles in a generic n-dimensional family of n-dimensional Lagrangian tori T^n. In particular, generic Lagrangian toric fibration $M \twoheadrightarrow \Delta \subset \mathbb{R}^n$ (i.e. a completely integrable system) should have a finite number of Bohr – Sommerfeld fibres.

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Planckian cycles and Berry bundle

A Planckian cycle $\widetilde{\varphi} : S \hookrightarrow P$ is a smoothly immersed Bohr – Sommerfeld cycle $\varphi : S \hookrightarrow M$ together with fixed horizontal unitary section of φ^*L, which can be viewed as *Legendrian embedding* $\widetilde{\varphi} : S \hookrightarrow P$ w.r.t. contact structure α on P:

$$\begin{tikzcd} & P \subset L \arrow[d, "p"] \\ S \arrow[ur, dashed, "\widetilde{\varphi}"] \arrow[r, "\varphi"'] & M . \end{tikzcd}$$

Space of Planckian cycles $\mathfrak{P}$ is a principal U_1-bundle over the space of Bohr – Sommerfeld cycles. It is called *Berry bundle* and denoted by $\pi : \mathfrak{P} \twoheadrightarrow \mathfrak{B}$.

Claim

Near a smooth embedding $\widetilde{\varphi} : S \hookrightarrow M$ the space of Planckian cycles $\mathfrak{P}$ is a smooth manifold locally modeled by some open neighborhood of zero in the vector space $C^\infty(S, \mathbb{R})$, of smooth real functions on S, which is canonically identified with the tangent space $T_{\widetilde{\varphi}}\mathfrak{P}$.

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Darboux – Weinstein uniformizations of $\mathfrak{P}$ and $\mathfrak{B}$

Extend underlying Bohr – Sommerfeld cycle $\varphi = p \circ \widetilde{\varphi} : S \to M$ to symplectic immersion $\widehat{\varphi} : \mathcal{U} \to M$ of a tube neighborhood $\mathcal{U} \subset T^*S$ of the zero section and write $\sigma : \mathcal{U} \hookrightarrow \widehat{\varphi}^* P$ for the unique unitary trivialization that coincides with $\widetilde{\varphi}$ over zero section and has the universal action form on T^*S as the connection 1-form: $\xi_\sigma = \sigma^* \widehat{\varphi}^* \alpha = \eta$. Restriction of $h\sigma \in \mathcal{C}^\infty(\mathcal{U}, \widehat{\varphi}^* P)$ onto Lagrangian cycle $s_\zeta : S \hookrightarrow \mathcal{U}$ (i.e. graph Γ_ζ of a closed 1-form $\zeta \in Z^1_{\mathrm{DR}}(S, \mathbb{R})$) is horizontal iff $d\ln(s_z^* h) = -i s_z^* \xi_\sigma = -i s_z^* \eta = -i\zeta$. Writing $s_z^* h = e^{-ig}$ for $g \in \mathcal{C}^\infty(S, \mathbb{R})$, we get $\zeta = dg$.

Darboux – Weinstein's coordinate on $\mathfrak{P}$ near $\widetilde{\varphi}$ takes smooth function $g \in \mathcal{C}^\infty(S, \mathbb{R})$ such that graph Γ_{dg} of exact 1-form dg lies in $\mathcal{U}$ to Bohr – Sommerfeld cycle $\widehat{\varphi}\left(\Gamma_{dg}\right)$ equipped with a horizontal unitary section $\sigma_g = e^{-ig} \sigma|_{\Gamma_{dg}}$. Near a smooth immersion $\varphi : S \hookrightarrow M$ the space of Bohr – Sommerfeld cycles $\mathfrak{B}$ is a smooth manifold locally modeled by open neighborhood of zero in the vector space of exact 1-forms on S, which is canonically identified with $T_\varphi \mathfrak{B}$. The differential of the Berry bundle $\pi : \mathfrak{P} \twoheadrightarrow \mathfrak{B}$ takes $g \mapsto dg$.

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Pseudo-holomorphic line bundles and the canonical class

Complex structure $I : TM \to TM$, $I^2 = -\mathrm{Id}$, leads to decomposition

$$\mathbb{C} \otimes TM = T_+M \oplus T_-M\,, \quad \text{where} \quad T_\pm M \overset{\text{def}}{=} \{v \in TM \mid Iv = \pm v\}\,.$$

Thus, we have *pseudo holomorphic functions* $f : M \to \mathbb{C}$, whose differentials annihilate T_-M, and *the canonical line bundle* $K = \Lambda^n T_+M$. Hermitian line bundle L on M inherits pseudo-holomorphic structure, which produces vector space of *pseudo holomorphic sections* $H^0_I(M,L)$.

Claim 1

The restriction of symplectic form ω onto the tangent cone C_xV to any pseudo holomorphic subvariety $V \subset M$ at any point $x \in V$ is non-degenerated: $\forall$ non-zero $v \in C_xV \quad \exists\, u \in C_xV : \omega(v,u) \neq 0$.

Claim 2

Non-zero pseudo holomorphic section of any pseudo holomorphic line bundle on M can not vanish identically along a Lagrangian cycle.

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Half-weights

Assume that $c_1(L) \sim c_1(K)$ are proportional and there exists a pseudo holomorphic line bundle $\mathcal{W} = \sqrt{L \otimes K}$ such that $\mathcal{W}^{\otimes 2} = L \otimes K$. Then $\varphi^*\mathcal{W}$ is topologically trivial for any Lagrangian immersion $\varphi : S \hookrightarrow M$ and each Planckian lift $\sigma : S \hookrightarrow \varphi^* P$ of φ produces non-degenerate pairing

$$\Omega_\sigma : \mathcal{C}^\infty(S, \varphi^*\mathcal{W}) \times \mathcal{C}^\infty(S, \varphi^*\mathcal{W}) \to \mathcal{C}^\infty(S, \varphi^* K)$$

defined by prescription $\mathcal{W} \otimes \mathcal{W} \ni \zeta_1 \otimes \zeta_2 = \sigma \otimes \Omega_s(\zeta_1, \zeta_2) \in L \otimes K$.

Definition

A half-weight on (S, σ) is a smooth global section $\varkappa : S \hookrightarrow \varphi^*\mathcal{W}$ such that $\mu(U) \stackrel{\text{def}}{=} \int_U \Omega_s(\varkappa, \varkappa)$ is a positive measure on M that belongs to the Radon-Nikodym equivalence class of measures provided by the structure of a smooth manifold on M.

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We write $\mathfrak{P}^{\mathrm{hw}}$ for the space of half-weighted Planckian cycles. Thus, a point of $\mathfrak{P}^{\mathrm{hw}}$ is a triple $(\varphi,\sigma,\varkappa)$, where

- $\varphi : S \hookrightarrow M$ is a Lagrangian immersion
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$\mathfrak{P}^{\mathrm{hw}}$ is locally modeled by a neighborhood of zero in the space $\mathcal{C}^\infty(S,\mathbb{C})$ of smooth functions $\psi = \psi_1 + i\psi_2$, $\psi_{1,2} \in \mathcal{C}^\infty(S,\mathbb{R})$. Complex Darboux – Weinstein coordinate near $(\varphi,\sigma,\varkappa)$ takes ψ to the graph $\Gamma_{d\psi_2}$ of exact 1-form $d\psi_2$ equipped with a horizontal unitary section $\sigma_{\psi_2} \stackrel{\text{def}}{=} e^{-i\psi_2}\sigma|_{\Gamma_{\psi_2}}$ and half-weighted by $\varkappa_{\psi_1} \stackrel{\text{def}}{=} e^{\psi_1}\varkappa$.

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Bortwick – Paul – Uribe map

Bortwick – Paul – Uribe map $\beta : \mathfrak{P}^{\mathrm{hw}} \to H^0(M,\mathcal{W})$ is composed from a map $\mathfrak{P}^{\mathrm{hw}} \to H^0(M,\mathcal{W})^*$, which takes half-weighted Planckian cycle $(\varphi,\sigma,\varkappa)$ to $\mathbb{C}$-linear functional $\beta_{(\varphi,\sigma,\varkappa)} : H^0(M,\mathcal{W}) \to \mathbb{C}$, $\varrho \mapsto \int_S \Omega_\sigma(\varkappa,\varrho)$, followed by $\mathbb{C}$-anti-linear isomorphism $H^0(M,\mathcal{W})^* \Rightarrow H^0(M,\mathcal{W})$ provided by the Hermitian form on $\mathcal{W}$. It has following remarkable properties:

- Being computed in the Darboux – Weinstein coordinates, its differential $d\beta : \mathcal{C}^\infty(S,\mathbb{C}) \to H^0(M,\mathcal{W})$ is $\mathbb{C}$-linear
- Hamiltonian reduction of β w.r.t. U_1-action leads to a map

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from the space of half-weighted Bohr – Sommerfeld cycles $\mathfrak{B}_t^{\mathrm{hw}}$ of fixed volume $t = \int_S \varkappa$ to the space of conformal blocks of the Kähler quantization of (M,ω).

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Moduli space of rank 2 bundles on a Riemann surface

Write Σ_J for a Riemann surface Σ of genus $g \geqslant 2$ equipped with (integrable) complex structure J. By the Narasihan – Seshadri – Donaldson theorem, associated with topologically trivial complex vector bundle $E \twoheadrightarrow \Sigma$ of $\operatorname{rk} E = 2$ is the moduli space $\mathfrak{M} = \mathfrak{M}(2, 0, \Sigma_J)$ that parametrizes geometric objects of 3 different types:

1. structures of stable holomorphic vector bundle $E \twoheadrightarrow \Sigma_J$ up to holomorphic isomorphism of holomorphic vector bundles
2. indecomposable flat Hermitian connections on E compatible with J up to $\mathcal{C}^{\infty}(\Sigma, \mathrm{SU}_2)$-gauge
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Prequantization equipment of the moduli space

The first description provides $\mathfrak{M}$ with integrable Kähler structure $I = I(J)$ depending on J. As a Kähler manifold $\mathfrak{M}$ has $\dim_{\mathbb{C}} \mathfrak{M} = 3g - 3$. Fibers of holomorphic tangent and cotangent bundles on $\mathfrak{M}$ at a given point E are

$$\mathcal{T}_E\mathfrak{M} = H^1(\Sigma, \mathrm{Ad}(E)), \quad \mathcal{T}^*_E\mathfrak{M} = \mathrm{Ad}(E, E \otimes K_\Sigma)$$

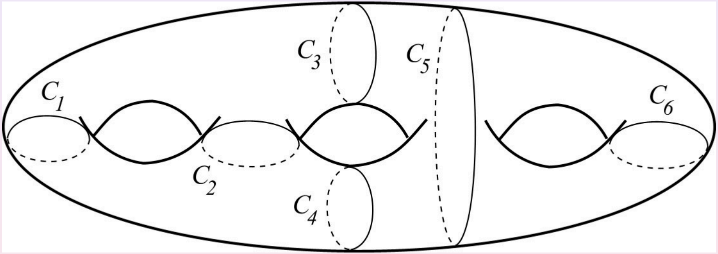
The Picard group $\mathrm{Pic}(\mathfrak{M}) = \mathbb{Z} \cdot \mathcal{L}$ is generated by ample holomorphic line bundle $\mathcal{L} \overset{\text{def}}{=} \mathcal{O}_{\mathfrak{M}}(\Theta)$ associated with *non-abelian theta-divisor*

$$\Theta \overset{\text{def}}{=} \left\{E \in \mathfrak{M} \mid H^0\left(\Sigma, E(g-1)\right) \neq 0\right\}.$$

The second description equips $\mathfrak{M}$ with symplectic structure induced via Hamiltonian reduction (w.r.t. the gauge group $\mathcal{C}^\infty(\Sigma, \mathrm{SU}_2)$ and the moment map provided by the curvature) from the canonical 2-form on the space of SU_2-connections: $\omega(\xi_1, \xi_2) = \int_\Sigma \mathrm{tr}(\xi_1 \wedge \xi_2)$, where $\xi_{1,2} \in \mathcal{C}^\infty(\Sigma, T^*\Sigma \otimes \mathfrak{su}_2)$ and tr means the contraction with the Killing form on $\mathfrak{su}_2$.

Goldman's real polarization

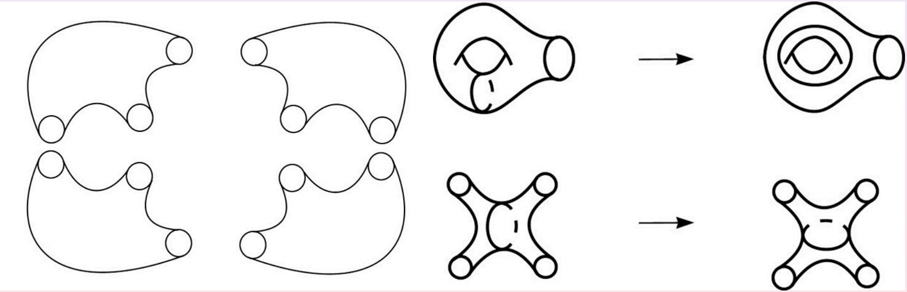
The third description leads to the completely integrable structure on $\mathfrak{M}$. Each simple loop $\gamma : S^1 \to \Sigma$ produces Hamiltonian $H_\gamma : \mathfrak{M} \to [0,1]$, $\varrho \mapsto \arccos\left(\mathrm{tr}(\varrho(\gamma))/2\right)$, where $\varrho : \pi_1(\Sigma) \to \mathrm{SU}_2$. Goldman has shown that $\{H_{\gamma_2}, H_{\gamma_2}\}_{\omega_{\mathfrak{M}}} = 0$ for non-isotopic non-intersecting simple loops $\gamma_{1,2} \in \pi_1(\Sigma)$.



Each collection $\gamma_1, \gamma_2, \ldots, \gamma_{3g-3}$ of $3g-3$ loops cutting Σ into pants produces Lagrangian fibration $\Gamma : \mathfrak{M} \twoheadrightarrow \Delta \subset [0,1]^{3g-3}$ whose Delzant polyhedron Δ is restricted by triangle inequalities $h_i - h_j \leqslant h_k \leqslant h_i + h_j$ written for all triples of loops $\gamma_i, \gamma_j, \gamma_k$ that do bound some pants.

Graph of pant decompositions

Complete systems of Goldman's Hamiltonians stay in bijection with pant decompositions and form the vertexes of graph whose edges are elementary regluings in pairs of adjacent pants



Generalized Knizhnik – Zamolodchikov correspondence

Bohr – Sommerfeld fibers of Goldman's fibration $\Gamma : \mathfrak{M} \twoheadrightarrow \Delta \subset [0,1]^{3g-3}$ w.r.t. pre – quantization bundle $L = \mathcal{L}^{\otimes k}$ are those lying over $\Delta \cap (\mathbb{Z}/k)^{3g-3}$. The Bortwick – Paul – Uribe map associates with each Bohr – Sommerfeld fiber a holomorphic section of $\mathcal{L}^{\otimes k}$ defined up to a constant factor. These sections form a basis in the space of conformal blocks $\mathbb{P}H^0(\mathfrak{M}, \mathcal{L})$.

When J runs through the moduli space $\mathcal{M}_g$ of algebraic curves of genus g conformal blocks $\mathbb{P}H^0(\mathfrak{M}, \mathcal{L})$ fill projective bundle $\mathcal{P} \twoheadrightarrow \mathcal{M}_g$. Each vertex of the graph of pants provides $\mathcal{P}$ with flat connection whose horizontal sections are those coming from Bohr – Sommerfeld fibers of Goldman's fibration.

Each $J \in \mathcal{M}_g$ equips the edges of the graph of pants with transition matrices between Bohr – Sommerfeld bases in $\mathbb{P}H^0(\mathfrak{M}, \mathcal{L})$ coming from Goldman's fibrations staying at the joined vertexes. These matrices produce «discrete field theory» of Wess – Zumino – Witten type.

THANKS FOR YOUR ATTENTION !