Modular differential operators and t-deformations of modular forms

Valery Gritsenko

Laboratoire Paul Painlevé, Lille
ILMSAF, NRU HSE, Moscow

Sirius Mathematics Center, Sochi, 27/02/2020

1. Motivations

The space of all versal deformations of the 14 exceptional Arnold's singularities is one parameter extension (*t-extension*) of type IV domains, i.e. $SO_0(2, n)/K$.

K. Saito's problem (1980-th): How one can extend orthogonal modular forms onto this non-classical homogeneous domain?

V. Gritsenko (2008): A non-trivial t-deformation exists for all modular forms except one example, the Borcherds Φ_{12}.

K. Saito: how can we extend possible *automorphic discriminants* of these exceptional singularities onto the corresponding t-domain? (This work is in progress.)

"Blow up of Cohen–Kuznetzov operator and an automorphic problem of K. Saito". Proc. of RIMS Symposium "Automorphic Representations, Automorphic Forms, L-functions, and Related Topics", Kokyuroki **1617** (2008), pp. 83–97.

2. t-deformation of the type IV domain and Saito's problem

Let L be a quadratic lattice of signature $(2, n)$ $(n \geq 3)$. The t-deformation of the homogeneous domain of $O(2, n)$ is

$$\mathcal{D}_L^t = \{[\mathbf{w}] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\mathbf{w}, \bar{\mathbf{w}}) > |(\mathbf{w}, \mathbf{w})| = t\}^+.$$

Definition. A *t-modular form* of weight k and character χ for an arithemtic subgroup $\Gamma < O^+(L)$ is a holomorphic function $F\colon (\mathcal{D}_L^t)^\bullet \to \mathbb{C}$ on the affine cone $(\mathcal{D}_L^t)^\bullet$ over $\mathcal{D}_L^t$ such that

$$F(\alpha v) = \alpha^{-k} F(v) \quad \forall\, \alpha \in \mathbb{C}^* \quad \text{and} \quad F(gv) = \chi(g) F(v) \quad \forall\, g \in \Gamma.$$

For $t = 0$ we have the type IV domain $\mathcal{D}_L$ and $O^+(L)$ forms.
Example: $w \in (\mathcal{D}_L^t)^\bullet$ is a modular form of weight -2.

Saito's problem. To construct a t-deformation of a $O^+(L)$-modular form of weight k.

3. The modular action on t

The tube realisation with a hyperbolic lattice $L_1 = u^{\perp}/\mathbb{Z}u$ ($u^2 = 0$)

$$\mathcal{H}^t = \mathcal{H}^t(L_1) = \{(Z;t) \in (L_1 \otimes \mathbb{C}) \times \mathbb{C} \mid (\operatorname{Im} Z, \operatorname{Im} Z) > \frac{|t| - \operatorname{Re} t}{2}\}^+$$

The relation with the projective model $\mathcal{D}_L^t$ is given by the following correspondence

$$(Z;t) \mapsto v = \begin{pmatrix} \frac{t-(Z,Z)}{2} \\ Z \\ 1 \end{pmatrix} \in \mathcal{D}_L^t, \qquad t = (w,w) \text{ if } w \in \mathcal{D}_L^t.$$

The fractional linear action of $O^+(L \otimes \mathbb{R})$ on the tube domain $\mathcal{H}^t$ and the automorphic factor $j(g;Z,t)$ of this action are defined as follows

$$g \cdot v = j(g;Z,t) \begin{pmatrix} \frac{t'-(Z',Z')}{2} \\ Z' \\ 1 \end{pmatrix} = j(g;Z,t) g\langle (Z,t) \rangle.$$

Theorem (**V. Gritsenko, 2008**) *For any modular form (except the Borcherds form Φ_{12}) there exists its non-trivial t-deformation.*

The case of $k > \frac{n_0}{2}$. Let $L = 2U \oplus L_0(-1)$ be a lattice of signature $(2, n_0+2)$ where $n_0 = \operatorname{rank} L_0 > 0$, $L_1 = U \oplus L_0(-1)$ and

$$F(Z) = \sum_{l \in L_1^*,\ (l,l) \geq 0} a(l) \exp(2\pi i (l, Z)) \in M_k(\tilde{O}^+(L), \chi).$$

Then

$$F(Z; t) = F(Z) + \sum_{l \in L_1^*} \sum_{\nu \geq 1} \frac{a(l)\,(l,l)^\nu\,(-\pi^2 t^2)^\nu}{(k - \frac{n_0}{2}) \dots (k - \frac{n_0}{2} + \nu - 1)\nu!} \exp(2\pi i (l, Z))$$

is a t-modular form of type $M_k^t(\tilde{O}^+(L), \chi)$.

5. $n_0 = -1$: t-deformation of Cohen–Kuznetsov–Zagier

Degeneration of Theorem for $n_0 = -1$. Jacobi type forms $J^t_{k,m}$: $\tau \in \mathbb{H}_1$, $t \in \mathbb{C}$,

$$\varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{t}{c\tau+d}\Big) = (c\tau+d)^k \exp\Big(2\pi i m \frac{ct^2}{c\tau+d}\Big)\varphi(\tau,t).$$

Let $f(\tau) \in M_k(SL_2(\mathbb{Z}))$. Then

$$\varphi_f(\tau,t) = \sum_{m=0}^{\infty} \frac{(2\pi i)^m (k-1)!}{m!(k+m-1)!} f^{(m)}(\tau)\, t^{2m}$$

is a Jacobi type form of weight k and index 1. This lifting gives the **generating function for the Rankin–Cohen brackets**:

$$\varphi_f(\tau,t)\varphi_g(\tau,-it) = \sum_{l\ge 0} [f(\tau), g(\tau)]_{2l}\, t^{2l}, \quad [f,g]_{2l} \in M_{k_f+k_g+2l}.$$

6. Algebra with two operators

In the ring $M_*[G_2]$ we fix two natural operators:

$$D,\ \ G_2\bullet : M_*[G_2] \to M_*[G_2].$$

$D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$ $(q = e^{2\pi i\tau})$ and multiplication by

$$G_2(\tau) = -D(\log(\eta(\tau))) = -\frac{1}{24} + \sum_{n\ge 1} \sigma_1(n)q^n.$$

In particular, $D(G_2) = -2G_2^2 + \frac{5}{6}G_4$. We have the quasi-modular operator

$$D_k = D + 2kG_2\bullet\ : M_k \to M_{k+2}$$

and its iterations

$$D_{k,n} = D_{k+2(n-1)} \circ \cdots \circ D_{k+2} \circ D_k : M_k \to M_{k+2n}.$$

7. The main part of the n-th iteration of D_k

Proposition. The major quasi-modular part $E_{k,n}$ of $D_{k,n}$ is given by the following sum

$$E_{k,n} = \sum_{\nu=0}^{n} \frac{n!\,\Gamma(k+n)}{\nu!(n-\nu)!\,\Gamma(k+\nu)}\,(2G_2)^{n-\nu} D^{\nu} : M_k \to M_{k+2n}.$$

(We use Γ-functions in the formulation in order to apply the same calculus in the case of negative or half integral weights.)

Proof. Using only one relation

$$D(G_2\bullet) \equiv -2G_2^2 \bullet + G_2 \cdot D \mod M_*, \qquad (1)$$

we obtain the proof

$$D_{k+2l}(E_{k,l}) = E_{k,l+1} + \frac{5}{3} G_4 \cdot E_{k,l-1} \equiv E_{k,l+1} \mod M_*.$$

8. Automorphic correction: Gritsenko, 1996

For $m = 0$ a Jacobi type form of index 0 is a formal power series over the rings of modular forms: $J^t_{k,0} = M_{k+*}[[t]]$. We can define the following operator of *automorphic correction* (Gritsenko, 1996)

$$\mathrm{AC}_m : J^t_{k,m} \to J^t_{k,0}$$

$$\mathrm{AC}_m : \varphi(\tau, t) \mapsto e^{-8\pi^2 m G_2(\tau) t^2} \varphi(\tau, t) = \sum_{n \geq 0} f_{k+n}(\tau) t^n \in J^t_{k,0}$$

where $f_{k+n}(\tau) \in M_{k+n}(SL_2(\mathbb{Z}))$. As a corollary of Proposition above we get Cohen–Kuznetsov–Zagier lifting:

$$\begin{array}{ccc} M_k & \xrightarrow{\nabla_E(X)} & JT_{k,0} \\ & \searrow^{\nabla_D(X)} & \downarrow{\scriptstyle e^{-2G_2 X}} \\ & & JT_{k,1}. \end{array}$$

In fact,

$$\nabla_E(X) = 1 + \sum_{n\geq 1} \frac{E_{k,n}}{n!\,\Gamma(k+n)} X^n = e^{2G_2 X}\nabla_D(X),$$

where

$$\nabla_D(X) = \sum_{\nu\geq 0} \frac{D^\nu}{\nu!\,\Gamma(k+\nu)} X^\nu.$$

If $X = -4\pi^2 mt^2$, then the last series defines the CKZ-operator from $M_k(SL_2(Z))$ to $J^t_{k,m}$

$$\nabla_D(X)(f) = \sum_{\nu\geq 0} \frac{D^\nu(f)}{\nu!\,\Gamma(k+\nu)} X^\nu \in J^t_{k,m}.$$

The same algebraic construction works for Jacobi modular forms!

Let $L_0 > 0$ be a positive definite even integral lattice.
Definition. A *Jacobi type form* of weight k and index m with parameter t (t^2 in the previous definition!) with respect to an even integral positive definite lattice L_0 is a holomorphic function $\phi(\tau, \mathfrak{z}; t)$ on $\mathbb{H}_1 \times (L_0 \otimes \mathbb{C}) \times \mathbb{C}$ which satisfies two equations

$$\phi(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}; \frac{t}{(c\tau + d)^2}) = (c\tau + d)^k \exp\left(\pi i m \frac{c(t + (\mathfrak{z}, \mathfrak{z}))}{c\tau + d}\right)\phi(\tau, \mathfrak{z}; t)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and for $\forall\, \lambda, \mu \in L_0$

$$\phi(\tau, \mathfrak{z} + \lambda\tau + \mu; t) = \exp\left(-\pi i m((\lambda, \lambda)\tau + 2(\lambda, \mathfrak{z}))\right)\phi(\tau, \mathfrak{z}; t).$$

For $t = 0$ one gets the Jacobi forms of the lattice index $L_0(m)$.

11. t-deformation with the heat operator

We put

$$H = 2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\omega} - S_0[\frac{\partial}{\partial\mathfrak{z}}], \quad G_2' = -8\pi^2 m G_2,$$

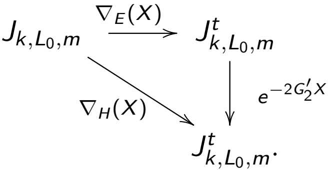
where S_0 is the Gram matrix of L_0. Then we have

$$H_k = H - 8\pi^2 m(2k - n_0)G_2\bullet \,:\, J_{k,m}(L_0) \to J_{k+2,m}(L_0).$$

We make the following changes in the previous algebraic structure:

$$D \mapsto H, \quad k \mapsto k - \frac{n_0}{2}, \quad G_2 \mapsto G_2' = -8\pi^2 m G_2.$$

Changing the structure constants in the previous proof we get a t-deformation of Jacobi modular forms $\phi(\tau,\mathfrak{z})\exp(2\pi i\omega) \in J_{k,L_0}$:



1) The algebraic method works for any modular form $f(\tau)$ or Jacobi modular form of negative, zero, half-integral weight or real weight.

2) The method gives CKZ-lifting of quasi-modular forms:

$$\nabla'_D(X)(G_2) = 1 - 2\sum_{\nu\geq 1} \frac{D^{\nu-1}(G_2)}{\nu!(\nu-1)!} X^\nu \in J^t_{0,m}, \ X = (2i\pi mz)^2.$$

3) The t-deformation gives interesting operator constructions for Siegel modular forms of genus 2 and for $SU(2,2)$ or $Sp(2,2)$ forms.

4) For the case of singular weight $k = \frac{n_0}{2}$ we have another construction of t-deformation. It gives a strange (quasi-modular) t-deformation of Siegel theta-series. It would be interesting to interpret this deformation in terms of a "t-deformed" (?) heat equation.