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Устойчивые турнирные решения как инструменты для принятия оптимальных решений: проблема обобщения двухпартийного множества на случай неполных предпочтений

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Alternatives, comparisons, choices

A – the *general set* of alternatives.

X – the *menu*: $X \subseteq A \wedge X \neq \varnothing \wedge |X| < \infty$.

R – results of binary comparisons, $R \subseteq A \times A$.

R is presumed to be complete: $\forall\, x \in A, \forall\, y \in A, (x, y) \in R \vee (y, x) \in R$.

$R|_X = R \cap X \times X$ – restriction of R onto X.

$(X, R|_X)$ – *abstract game or weak tournament*.

P – asymmetric part of R, $P \subseteq R$: $(x, y) \in P \Leftrightarrow ((x, y) \in R \wedge (y, x) \notin R)$.

If $P|_X$ is complete, $\forall\, x \in A, \forall\, y \in A \wedge y \neq x, (x, y) \in P \vee (y, x) \in P$, then

$(X, R|_X)$ – (*proper*) *tournament*.

Tournament solutions

A *tournament solution S* is a correspondence

$$S(X, R): 2^A \backslash \varnothing \times 2^{A\times A} \rightarrow 2^A$$

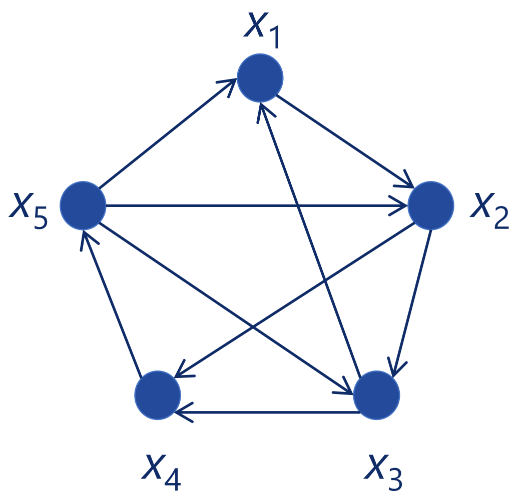
that has the following properties:

0. *Locality*: $S(X, R)=S(R|_X) \subseteq X$
1. *Nonemptiness*: $\forall X, \forall R, \ S(R|_X) \neq \varnothing$;
2. *Neutrality*: permutation of alternatives' names and choice commute;
3. *Condorcet consistency*:

$$MAX(R|_X) \subseteq S(R|_X) \wedge MAX(R|_X)=\{w\} \Rightarrow S(R|_X)=\{w\}.$$

	x_1	x_2	x_3	x_4	x_5
x_1	0	1	0	0	0
x_2	0	0	1	1	0
x_3	1	0	0	1	0
x_4	0	0	0	0	1
x_5	1	1	1	0	0

Tournament matrix **T**



Tournament digraph

Lotteries

Comparison function: $g(x_1, x_2)=1 \Leftrightarrow x_1 P x_2$, $g(x_1, x_2)=-1 \Leftrightarrow x_2 P x_1$, otherwise $g(x_1, x_2)=0$.

	x_1	x_2	x_3
x_1	0	1	0
x_2	0	0	1
x_3	1	0	0

Tournament matrix **T**

Since matrix $\mathbf{G} = \|g(x_i, x_j)\|$ is skew-symmetric,

formula $\mathbf{p}_1\mathbf{G}\mathbf{p}_2$ defines a binary relation on the set of lotteries: $\mathbf{p}_1\mathbf{G}\mathbf{p}_2 \geq 0 \Leftrightarrow \mathbf{p}_1 \succsim \mathbf{p}_2$.

If $\mathbf{p}_0\mathbf{G}\mathbf{p} \geq 0$ for all $\mathbf{p}$ then $\mathbf{p}_0$ is a *maximal lottery*.

The set $\{x\}$ is the support of a maximal lottery on $X \Leftrightarrow x$ is a maximal element of $R|_X$.

	x_1	x_2	x_3
x_1	0	1	-1
x_2	-1	0	1
x_3	1	-1	0

Matrix **G**

Bipartisan set (*BP*) and Essential set (*E*)

1. The set of maximal lotteries is always nonempty.
2. If a tournament $(X, R|_X)$ is proper then there is just one maximal lottery.

Bipartisan set *BP* (Laffond, Laslier, Le Breton, 1993)

of a (proper) tournament $(X, R|_X)$ is the support of the (unique) maximal lottery.

Essential set *E* (Dutta, Laslier, 1999)

of a (weak) tournament $(X, R|_X)$ is the union of supports of all maximal lotteries.

Example

The Condorcet cycle.

$X=\{x_1, x_2, x_3\}$, $R|_X=\{(x_1, x_2), (x_1, x_2), (x_1, x_2)\}$.

Maximal lottery $\mathbf{p}_{max}$= (1/3, 1/3, 1/3).

Bipartisan set $BP = X$.

Note that $\mathbf{p}_{max}$ is an eigenvector of **G** with the eigenvalue 0,

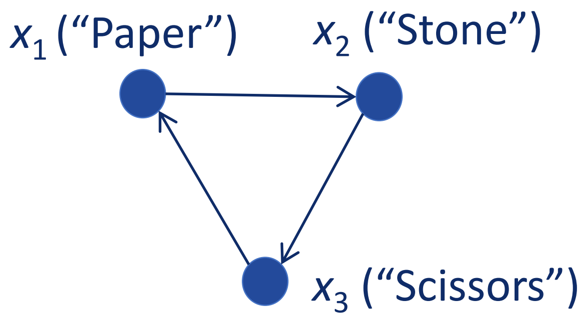
therefore $\mathbf{pGp}_{max}=0$ for all **p**.

	x_1	x_2	x_3
x_1	0	1	0
x_2	0	0	1
x_3	1	0	0

Tournament matrix **T**

	x_1	x_2	x_3
x_1	0	1	-1
x_2	-1	0	1
x_3	1	-1	0

Matrix **G**



Tournament game –
“Paper, Scissors, Stone”

Properties

- ***Monotonicity***

$$\forall R_1, R_2 \subseteq A^2,\ \forall X \subseteq A,\ \forall x \in S(R_1|_X),$$

$$(R_1|_{X\setminus\{x\}} = R_2|_{X\setminus\{x\}} \wedge \forall y \in X\setminus\{x\},\ (xP_1y \Rightarrow xP_2y) \wedge (xR_1y \Rightarrow xR_2y)) \Rightarrow x \in S(R_2|_X).$$

- ***Stability***

For all $R \subseteq A^2$ and for all $X, Y \subseteq A$ such that $X \cap Y \neq \varnothing$ the following holds:

$$S(X, R) = S(Y, R) = Z \Leftrightarrow S(X \cup Y, R) = Z \wedge Z \subseteq X \cap Y.$$

- ***Computational simplicity***

There is a polynomial algorithm for computing S.

Properties related to stability

Stability: $S(X, R)=S(Y, R)=Z \Leftrightarrow S(X\cup Y, R)=Z \wedge Z \subseteq X\cap Y$.

- ***α-property*** (*Outcast property or generalized Nash independence of irrelevant alternatives*):

$$S(X, R)=S(Y, R)=Z \Leftarrow S(X\cup Y, R)=Z \wedge Z \subseteq X\cap Y.$$

- ***γ-property***:

$$S(X, R)=S(Y, R)=Z \Rightarrow S(X\cup Y, R)=Z \wedge Z \subseteq X\cap Y.$$

- ***Idempotence***: $\forall X, S(S(X))=S(X)$.
- ***The Aizerman-Aleskerov condition***: $\forall X, \forall Y, S(X)\subseteq Y\subseteq X \Rightarrow S(Y)\subseteq S(X)$.
- ***Independence of irrelevant results*** (*independence of losers)*:

$$\forall R_1,R_2\subseteq A^2, \forall X\subseteq A, (\forall x\in S(R_1|_X),\forall y\in X, ((xR_1y \Leftrightarrow xR_2y) \wedge (yR_1x \Leftrightarrow yR_2x)) \Rightarrow S(R_1|_X)=S(R_2|_X).$$

α-property ⟺ Idempotence ∧ the Aizerman-Aleskerov condition

α-property ⟹ Independence of irrelevant results

The conservative extension (Brandt et al., 2014, 2018)

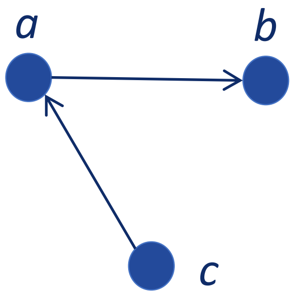
A tournament (X, T) is called ***orientation*** of a weak tournament (X, P) if (X, T) is proper and $P \subseteq T$.

For a tournament solution S(X, P), its ***conservative extension*** (denoted [S]) to weak tournaments is the choice correspondence [S](X, P) defined thus:

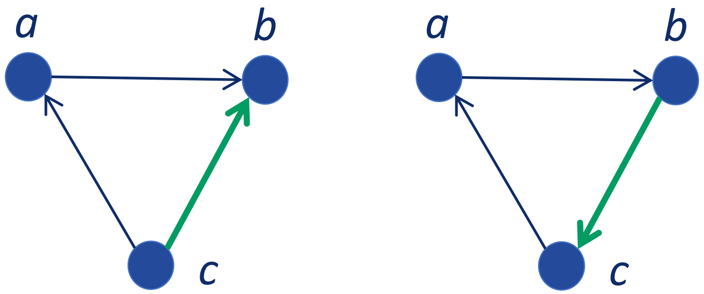
an alternative x from X belongs to [S](X, P) if and only if there is an ***orientation*** (X, T) of (X, P), such that x belong to S(X, T).

Theorem: The conservative extension preserves properties of the original solution.

Example



Weak tournament digraph



Orientations of the weak tournament

$[BP]$=?

$BP=\{c\}$

$BP=\{a, b, c\}$

$[BP] =\{a, b, c\}$

Axiomatic analysis

	BP	[*BP*]	*E*
Monotonicity	Yes	Yes	Yes
α-property (outcast)	Yes	Yes	Yes
Idempotence	Yes	Yes	Yes
Aizerman-Aleskerov property	Yes	Yes	Yes
Independence of irrelevant results	Yes	Yes	Yes
γ-property	Yes	Yes	Yes
Stability	Yes	Yes	Yes
Computational simplicity	Yes	Yes	Yes

Relations of *E* and [*BP*] to other solutions

In proper tournaments, $E=[BP]=BP \subseteq UC \subseteq ES$, also $BP \subseteq D \subseteq ES$.

In weak tournaments,

	E	$[BP]$	UC_{IM}	UC_M	UC_{IF}	UC_F	UC_{McK}	UC_D	D	SP	$[D]$	WS	ES	RES	$[ES]$	UCp	STC	UT	WTC
E	=	∩	∅	∅	∅	∅	⊂	⊂	∅	∩	⊂	∩	∩	∩	⊂	⊂	∩	∩	⊂
$[BP]$	∩	=	∩	∩	∩	∩	∩	⊂	∩	∩	⊂	∩	∩	∩	⊂	∩	∩	∩	⊂

If S_1 and S_2 are the tournament solutions denoting, correspondingly, a row and a column, then a symbol in the corresponding box means the following:

"∅" – S_1 and S_2 are independent, that is, their intersection can be empty in some finite tournament;

"∩" – S_1 and S_2 are not logically nested, but their intersection is always nonempty;

"⊂" – S_1 refines S_2; "=" – S_1 and S_2 are identical.

E and [*BP*] are not logically nested (Brandt et al., 2018)

We proved that *E* and [*BP*] are not independent: $E \cap [BP] \neq \varnothing$ always.

Lemma (Tucker 1956): For any skew-symmetric matrix $\mathbf{G}$ there exists a vector $\mathbf{p}$ such that

$\mathbf{p} \geq \mathbf{0}$ and $\mathbf{Gp} \geq \mathbf{0}$ and $\mathbf{pGp} = 0$ and $\mathbf{p} + \mathbf{Gp} > \mathbf{0}$.

Restatement of Tucker's lemma

For any antisymmetric comparison function $g(x, y): X \times X \to \mathbb{R}$ there exists a lottery **p** on *X*

such that $\forall y \in X, \sum_{x \in X} p(x) g(x, y) \geq 0$

and exactly one of the two numbers $p(y)$ and $\sum_{x \in X} p(x) g(x, y)$ is positive, while the other is 0.

Спасибо за внимание!

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The covering relations and the uncovered sets

The covering relations (Fishburn, 1977; Miller, 1980; McKelvey, 1986; Duggan, 2007, 2013)

The covering relation $C \subseteq X^2$, is a strengthening of $P|_X$:

1. The Miller covering C_M: $xC_My \Leftrightarrow xPy \wedge P^{-1}(y) \subset P^{-1}(x)$.
2. The weak Miller covering C_{WM}: $xC_{WM}y \Leftrightarrow P^{-1}(y) \subset P^{-1}(x)$.
3. The Fishburn covering C_F: $xC_Fy \Leftrightarrow xPy \wedge P(x) \subset P(y)$.
4. The weak Fishburn covering C_{WF}: $xC_{WF}y \Leftrightarrow P(x) \subset P(y)$.
5. The McKelvey covering C_{McK}: $xC_{McK}y \Leftrightarrow xPy \wedge P^{-1}(y) \subset P^{-1}(x) \wedge P(x) \subset P(y)$.
6. The Duggan covering C_D: $xC_Dy \Leftrightarrow P^0(y) \cup P^{-1}(y) \subset P^{-1}(x)$.

The set of all alternatives that are not (weakly) covered in X by any alternative is called ***the (inner) uncovered set*** of a feasible set X.

The Miller, Fishburn, McKelvey and Duggan uncovered sets and their inner versions will be denoted UC_M, UC_F, UC_{McK}, UC_D, UC_{IM} and UC_{IF}, correspondingly.

Minimal externally stable sets

A nonempty subset Y of X is called

P-dominating if $\forall x \in X, \exists y \in Y: yPx$

P-externally stable if $\forall x \in X \backslash Y, \exists y \in Y: yPx$

R-externally stable if $\forall x \in X \backslash Y, \exists y \in Y: yRx$

Self-protecting if $\forall x \in X, (\exists y \in Y: yPx) \vee (\forall y \in Y, yRx)$

Weakly stable if $\forall x \in X \backslash Y, (\exists y \in Y: yPx) \vee (\forall y \in Y, yRx)$

Tournament solutions: the union of all minimal

P-dominating sets *D* (Duggan 2013, Subochev 2016)

P-externally stable sets *ES* (Wuffl, Feld, Owen & Grofman 1989, Subochev 2008)

R-externally stable sets *RES* (Aleskerov & Subochev 2009, 2013)

Self-protecting sets *SP* (Roth 1976, Subochev 2020)

Weakly stable sets *WS* (Aleskerov & Kurbanov 1999)